

## STEENROD COALGEBRAS III. THE FUNDAMENTAL GROUP

JUSTIN R. SMITH

ABSTRACT. In this note, we extend earlier work by showing that if  $X$  and  $Y$  are delta-complexes (i.e. simplicial sets without degeneracy operators), a morphism  $g: N(X) \rightarrow N(Y)$  of Steenrod coalgebras (normalized chain-complexes equipped with extra structure) induces one of 2-skeleta  $\hat{g}: X_2 \rightarrow Y_2$ , inducing a homomorphism  $\pi_1(\hat{g}): \pi_1(X) \rightarrow \pi_1(Y)$  that is an isomorphism if  $g$  is an isomorphism. This implies a corresponding conclusion for a morphism  $g: C(X) \rightarrow C(Y)$  of Steenrod coalgebras on *unnormalized* chain-complexes of *simplicial sets*.

## 1. INTRODUCTION

It is well-known that the Alexander-Whitney coproduct is functorial with respect to simplicial maps. If  $X$  is a simplicial set,  $C(X)$  is the unnormalized chain-complex and  $RS_2$  is the *bar-resolution* of  $\mathbb{Z}_2$  (see [1]), it is also well-known that there is a unique homotopy class of  $\mathbb{Z}_2$ -equivariant maps (where  $\mathbb{Z}_2$  transposes the factors of the target)

$$\xi_X: RS_2 \otimes C(X) \rightarrow C(X) \otimes C(X)$$

cohomology, and that this extends the Alexander-Whitney diagonal. We will call such structures, Steenrod coalgebras and the map  $\xi_X$  the Steenrod diagonal.

With some care (see appendix A of [3]), one can construct  $\xi_X$  in a manner that makes it *functorial* with respect to simplicial maps although this is seldom done since the *homotopy class* of this map is what is generally studied. The paper [3] showed that:

**Corollary. 3.8.** *If  $X$  and  $Y$  are simplicial complexes (simplicial sets without degeneracies whose simplices are uniquely determined by their vertices), any purely algebraic chain map of normalized chain complexes*

$$f: N(X) \rightarrow N(Y)$$

---

2000 *Mathematics Subject Classification.* Primary 18G55; Secondary 55U40.  
*Key words and phrases.* operads, cofree coalgebras.

that makes the diagram

$$(1.1) \quad \begin{array}{ccc} RS_2 \otimes N(X) & \xrightarrow{1 \otimes f} & RS_2 \otimes N(Y) \\ \xi_X \downarrow & & \downarrow \xi_Y \\ N(X) \otimes N(X) & \xrightarrow{f \otimes f} & N(Y) \otimes N(Y) \end{array}$$

commute induces a map of simplicial complexes

$$\hat{f}: X \rightarrow Y$$

If  $f$  is an isomorphism then  $\hat{f}$  is an isomorphism of simplicial complexes — and  $X$  and  $Y$  are homeomorphic.

The note extends that result, slightly, to

**Corollary. 3.8** *If  $X$  and  $Y$  are delta-complexes, any morphism of their canonical Steenrod coalgebras (see proposition 3.2)*

$$g: N(X) \rightarrow N(Y)$$

induces a map

$$\hat{g}: X_2 \rightarrow Y_2$$

of 2-skeleta. If  $g$  is an isomorphism then  $X_2$  and  $Y_2$  are isomorphic as delta-complexes.

and

**Corollary. 3.9** *If  $X$  and  $Y$  are simplicial sets and  $f: C(X) \rightarrow C(Y)$  is a morphism of their canonical Steenrod coalgebras (see proposition 3.2) over their unnormalized chain-complexes, then  $f$  induces a map*

$$\hat{f}: X_2 \rightarrow Y_2$$

of 2-skeleta. If  $f$  is an isomorphism, then  $\hat{f}$  is a homotopy equivalence.

The author conjectures that the last statement can be improved to “if  $f$  is an isomorphism, then  $\hat{f}$  is a homotopy equivalence.”

The author is indebted to Dennis Sullivan for several interesting discussions.

## 2. DEFINITIONS AND ASSUMPTIONS

Given a simplicial set,  $X$ ,  $C(X)$  will always denote its *unnormalized* chain-complex and  $N(X)$  its *normalized* one (with degeneracies divided out).

We consider variations on the concept of simplicial set.

**Definition 2.1.** Let  $\Delta_+$  be the ordinal number category whose morphisms are order-preserving monomorphisms between them. The objects of  $\Delta_+$  are elements  $\mathbf{n} = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$  and a morphism

$$\theta: \mathbf{m} \rightarrow \mathbf{n}$$

is a strict order-preserving map ( $i < k \implies \theta(i) < \theta(j)$ ). Then the category of *delta-complexes*,  $\mathbf{D}$ , has objects that are contravariant functors

$$\Delta_+ \rightarrow \mathbf{Set}$$

to the category of sets. The chain complex of a delta-complex,  $X$ , will be denoted  $N(X)$ .

*Remark.* In other words, delta-complexes are just simplicial sets *without degeneracies*.

A simplicial set gives rise to a delta-complex by “forgetting” its degeneracies — “promoting” its degenerate simplices to nondegenerate status. Conversely, a delta-complex can be converted into a simplicial set by equipping it with degenerate simplices in a mechanical fashion. These operations define functors:

**Definition 2.2.** The functor

$$\mathfrak{f}: \mathbf{S} \rightarrow \mathbf{D}$$

is defined to simply drop degeneracy operators (degenerate simplices become nondegenerate). The functor

$$\mathfrak{d}: \mathbf{D} \rightarrow \mathbf{S}$$

equips a delta complex,  $X$ , with degenerate simplicies and operators via

$$(2.1) \quad \mathfrak{d}(X)_m = \bigsqcup_{\mathbf{m} \twoheadrightarrow \mathbf{n}} X_n$$

for all  $m > n \geq 0$ .

*Remark.* The functors  $\mathfrak{f}$  and  $\mathfrak{d}$  were denoted  $F$  and  $G$ , respectively, in [2]. Equation 2.1 simply states that we add all possible degeneracies of simplices in  $X$  subject *only* to the basic identities that face- and degeneracy-operators must satisfy.

Although  $\mathfrak{f}$  promotes degenerate simplicies to nondegenerate ones, these new nondegenerate simplices can be collapsed without changing the homotopy type of the complex: although the degeneracy operators are no longer built in to the delta-complex, they still define contracting homotopies.

The definition immediately implies that

**Proposition 2.3.** *If  $X$  is a simplicial set and  $Y$  is a delta-complex,  $C(X) = N(\mathfrak{f}(X))$ ,  $N(\mathfrak{d}(Y)) = N(Y)$ , and  $C(X) = N(\mathfrak{d} \circ \mathfrak{f}(X))$ .*

Theorem 1.7 of [2] shows that there exists an adjunction:

$$(2.2) \quad \mathfrak{d}: \mathbf{D} \leftrightarrow \mathbf{S}: \mathfrak{f}$$

The composite (the *counit* of the adjunction)

$$\mathfrak{f} \circ \mathfrak{d}: \mathbf{D} \rightarrow \mathbf{D}$$

maps a delta complex into a much larger one — that has an infinite number of (degenerate) simplices added to it. There is a natural inclusion

$$\iota: X \rightarrow \mathfrak{f} \circ \mathfrak{d}(X)$$

and a natural map (the *unit* of the adjunction)

$$(2.3) \quad g: \mathfrak{d} \circ \mathfrak{f}(X) \rightarrow X$$

The functor  $g$  sends degenerate simplices of  $X$  that had been “promoted to nondegenerate status” by  $\mathfrak{f}$  to their degenerate originals — and the extra degenerates added by  $\mathfrak{d}$  to suitable degeneracies of the simplices of  $X$ .

In [2], Rourke and Sanderson also prove:

**Proposition 2.4.** *If  $X$  is a simplicial set and  $Y$  is a delta-complex then*

- (1)  $|Y|$  and  $|\mathfrak{d}Y|$  are homeomorphic
- (2) the map  $|g|: |\mathfrak{d} \circ \mathfrak{f}(X)| \rightarrow |X|$  is a homotopy equivalence.
- (3)  $\mathfrak{f}: HS \rightarrow HD$  defines an equivalence of categories, where  $HS$  and  $HD$  are the homotopy categories, respectively, of  $\mathbf{S}$  and  $\mathbf{D}$ . The inverse is  $\mathfrak{d}: HD \rightarrow HS$ . In particular, if  $X$  is a simplicial set, the natural map

$$g: \mathfrak{d} \circ \mathfrak{f}(X) \rightarrow X$$

*is a homotopy equivalence.*

*Remark.* Here,  $|*|$  denotes the topological realization functors for  $\mathbf{S}$  and  $\mathbf{D}$ .

*Proof.* The first two statements are proposition 2.1 of [2] and statement 3 is theorem 6.9 of the same paper. The final statement follows from Whitehead’s theorem.  $\square$

## 3. STEENROD COALGEBRAS

We begin with:

**Definition 3.1.** A *Steenrod coalgebra*,  $(C, \delta)$  is a chain-complex  $C \in \mathbf{Ch}$  equipped with a  $\mathbb{Z}_2$ -equivariant chain-map

$$\delta: RS_2 \otimes C \rightarrow C \otimes C$$

where  $\mathbb{Z}_2$  acts on  $C \otimes C$  by swapping factors and  $RS_2$  is the bar-resolution of  $\mathbb{Z}$  over  $\mathbb{Z}S_2$ . A morphism  $f: (C, \delta_C) \rightarrow (D, \delta_D)$  is a chain-map  $f: C \rightarrow D$  that makes the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \delta_C \downarrow & & \downarrow \delta_D \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array}$$

commute.

Steenrod coalgebras are very general — the underlying coalgebra need not even be coassociative. The category of Steenrod coalgebras is denoted  $\mathcal{S}$ .

Appendix A of [3] shows that:

**Proposition 3.2.** *If  $X$  is a simplicial set or delta-complex, then the unnormalized and normalized chain-complexes of  $X$  have a natural Steenrod coalgebra structure, i.e. natural maps*

$$\begin{aligned} \xi: RS_2 \otimes N(X) &\rightarrow N(X) \otimes N(X) \\ \xi: RS_2 \otimes C(X) &\rightarrow C(X) \otimes C(X) \end{aligned}$$

*Remark.* If  $[]$  is the 0-dimensional generator of  $RS_2$ , the map  $\xi([] \otimes *): N(X) \rightarrow N(X) \otimes N(X)$  is nothing but the Alexander-Whitney coproduct.

The Steenrod coalgebra structure for  $N(X)$  is a *natural quotient* of that for  $C(X)$ .

Here are some computations of this Steenrod coalgebra structure from appendix A of [3]:

**Fact.** *If  $\Delta^2$  is a 2-simplex, then*

$$(3.1) \quad \xi([] \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2$$

— the standard (Alexander-Whitney) coproduct — and

$$(3.2) \quad \begin{aligned} \xi([(1, 2)] \otimes \Delta^2) &= \Delta^2 \otimes F_0 \Delta^2 - F_1 \Delta^2 \otimes \Delta^2 \\ &\quad - \Delta^2 \otimes F_2 \Delta^2 \end{aligned}$$

Corollary 4.3 of [3] proves that:

**Corollary 3.3.** *Let  $X$  be a simplicial set and suppose*

$$f: \mathcal{N}^n = N(\Delta^n) \rightarrow N(X)$$

*is a Steenrod coalgebra morphism. Then the image of the generator  $\Delta^n \in N(\Delta^n)_n$  is a generator of  $N(X)_n$  defined by an  $n$ -simplex of  $X$ .*

We can prove a delta-complex (partial) analogue of corollary 4.5 in [3]:

**Corollary 3.4.** *Let  $X$  be a delta-complex, let  $n \leq 2$ , and let*

$$f: N(\Delta^n) \rightarrow N(X)$$

*map  $\Delta^n$  to a simplex  $\sigma \in N(X)$  defined by the simplicial-map  $\iota: \Delta^n \rightarrow X$ . Then  $f = N(\iota)$ .*

*Proof.* Let

$$\xi_i = \xi(e_i \otimes *): N(\Delta^n) \rightarrow N(\Delta^n) \otimes N(\Delta^n)$$

denote the Steenrod coalgebra structure, where  $e_i$  is the generator of  $(RS_2)_i$ . By hypothesis, the diagram

$$\begin{array}{ccc} N(\Delta^n) & \xrightarrow{1 \otimes f} & N(X) \\ \xi_i \downarrow & & \downarrow \xi_i \\ N(\Delta^n) \otimes N(\Delta^n) & \xrightarrow{f \otimes f} & N(X) \otimes N(X) \end{array}$$

commutes for all  $i \geq 0$ .

If  $\iota$  is an inclusion (and  $n$  is arbitrary), the conclusion follows from corollary 4.5 in [3]. If  $n = 1$ , and  $\iota$  identifies the endpoints of  $\Delta^1$ , there is a *unique* morphism from  $N(\Delta^1)$  to  $\text{im } N(\iota)$  that sends  $N(\Delta^1)_1$  to  $\text{im } N(\iota)_1$ .

If  $n = 2$ , equation 3.1 implies that

$$\text{im}(\xi_0(\Delta^2)) = F_2 \Delta^2 \otimes F_0 \Delta^2 \in (N(X)/N(X)_0) \otimes (N(X)/N(X)_0)$$

Since corollary 3.4 implies that  $f(\Delta^2)_2 = N(\iota)(\Delta^2)_2$ , it follows that the Steenrod-coalgebra morphism,  $f$ , must send  $F_i \Delta^2$  to  $N(\iota)(F_i \Delta^2)$  for  $i = 0, 2$ .

Equation 3.2 implies that

$$\text{im}(\xi_1(\Delta^2)) = -F_1\Delta^2 \otimes \Delta^2 \in N(X)_1 \otimes (N(X)/N(X)_1)$$

so that  $f(F_1\Delta^2) = N(\iota)(F_1\Delta^2)$  as well.  $\square$

We define a complement to the  $N(*)$ -functor:

**Definition 3.5.** Define a functor

$$\text{hom}_{\mathcal{S}}(\star, *): \mathcal{S} \rightarrow \mathbf{D}$$

to the category of delta-complexes (see definition 2.1), as follows:

If  $C \in \mathcal{S}$ , define the  $n$ -simplices of  $\text{hom}_{\mathcal{S}}(\star, C)$  to be the Steenrod coalgebra morphisms

$$\mathcal{N}^n \rightarrow C$$

where  $\mathcal{N}^n = N(\Delta^n)$  is the normalized chain-complex of the standard  $n$ -simplex, equipped with the Steenrod coalgebra structure defined in

Face-operations are duals of coface-operations

$$d_i: [0, \dots, i-1, i+1, \dots, n] \rightarrow [0, \dots, n]$$

with  $i = 0, \dots, n$  and vertex  $i$  in the target is *not* in the image of  $d_i$ .

**Proposition 3.6.** *If  $X$  is a delta-complex there exists a natural inclusion*

$$u_X: X \rightarrow \text{hom}_{\mathcal{S}}(\star, N(X))$$

*Remark.* This is also true if  $X$  is an arbitrary simplicial set.

*Proof.* To prove the first statement, note that any simplex  $\Delta^k$  in  $X$  comes equipped with a map

$$\iota: \Delta^k \rightarrow X$$

The corresponding order-preserving map of vertices induces an Steenrod-coalgebra morphism

$$N(\iota): N(\Delta^k) = \mathcal{N}^k \rightarrow N(X)$$

so  $u_X$  is defined by

$$\Delta^k \mapsto N(\iota)$$

It is not hard to see that this operation respects face-operations.  $\square$

So,  $\text{hom}_{\mathcal{S}}(\star, N(X))$  naturally contains a copy of  $X$ . The interesting question is whether it contains *more* than  $X$ :

**Theorem 3.7.** *If  $X \in \mathbf{D}$  is a delta-complex then the canonical inclusion*

$$u_X: X \rightarrow \text{hom}_{\mathcal{S}}(\star, N(X))$$

*defined in proposition 3.6 is the identity map on 2-skeleta.*

*Proof.* This follows immediately from corollary 3.3, which implies that simplices map to simplices and corollary 3.4, which implies that these maps are *unique*.  $\square$

**Corollary 3.8.** *If  $X$  and  $Y$  are delta-complexes, any morphism of their canonical Steenrod coalgebras (see proposition 3.2)*

$$g: N(X) \rightarrow N(Y)$$

*induces a map*

$$\hat{g}: X_2 \rightarrow Y_2$$

*of 2-skeleta. If  $g$  is an isomorphism then  $X_2$  and  $Y_2$  are isomorphic as delta-complexes.*

*Proof.* Any morphism  $g: N(X) \rightarrow N(Y)$  induces a morphism of simplicial sets

$$\text{hom}(\star, g): \text{hom}_{\mathcal{S}}(\star, N(X)) \rightarrow \text{hom}_{\mathcal{S}}(\star, N(Y))$$

which is an isomorphism (and homeomorphism) of simplicial complexes if  $g$  is an isomorphism. The conclusion follows from theorem 3.7 which implies that  $X_2 = \text{hom}(\star, N(X))_2$  and  $Y_2 = \text{hom}(\star, N(Y))_2$ .  $\square$

Propositions 2.3 and 2.4 imply that

**Corollary 3.9.** *If  $X$  and  $Y$  are simplicial sets and  $f: C(X) \rightarrow C(Y)$  is a morphism of their canonical Steenrod coalgebras (see proposition 3.2) over their unnormalized chain-complexes, then  $f$  induces a map*

$$\hat{f}: X_2 \rightarrow Y_2$$

*of 2-skeleta. If  $f$  is an isomorphism, then  $\hat{f}$  is a homotopy equivalence.*

*Proof.* Simply apply corollary 3.8 to  $\mathfrak{f}(X)$  and  $\mathfrak{f}(Y)$  and then apply  $\mathfrak{d}$  and proposition 2.4 to the map

$$\hat{f}: \mathfrak{f}(X)_2 \rightarrow \mathfrak{f}(Y)_2$$

that results.  $\square$

## REFERENCES

1. S. MacLane, *Homology*, Springer-Verlag, 1995.
2. C. P. Rourke and B. J. Sanderson,  $\Delta$ -sets. I. *Homotopy theory*, Quart. J. Math. Oxford **22** (1971), 321–338.
3. Justin R. Smith, *Steenrod coalgebras II. Simplicial complexes*, arXiv:1402.3134 [math.AT].

*Current address:* Department of Mathematics, Drexel University, Philadelphia, PA 19104

*E-mail address:* jsmith@drexel.edu

*URL:* <http://vorpai.math.drexel.edu>